

A PASCAL-LIKE TRIANGLE RELATED TO THE TRIBONACCI NUMBERS

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ABSTRACT. A tile composed of two pieces, which we refer to as a fence tile, is introduced to give the Tribonacci numbers a new combinatorial interpretation. The interpretation is used to construct a Pascal-like triangle and various identities concerning the triangle are proven. An intuitive proof of a general identity for the Tribonacci numbers in terms of sums of products of the binomial coefficients is given. A connection with sequence A120415 from Sloane's Encyclopedia of Integer Sequences is also made.

1. INTRODUCTION

Tiling interpretations of the Fibonacci numbers are presented in [3–6]. In [1], Benjamin explores many properties of the Lucas triangle using the bracelet tiling interpretation of the Lucas numbers. A generalization of the Fibonacci sequence, the Tribonacci sequence, is defined as follows. For $n \geq 3$, let

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \quad T_0 = 0, T_1 = 1, T_2 = 1.$$

From [2], it follows that if we let t_n be the number of ways to tile a $1 \times n$ rectangle (called an n -board with cells labeled $j = 1, 2, 3, \dots, n$) with squares, dominoes, and triominoes, then $t_n = T_{n+1}$. A tiling of an n -board is known as an n -tiling of which this is an example.

In this paper, we introduce a new type of tile, which we call a fence tile, allowing us to formulate a new tiling interpretation of the Tribonacci numbers. The number of n -tilings using a specified number of fence tiles is then used to create entries in a Pascal-like triangle. Intuitive combinatorial proofs of several of the triangle's properties are given.

2. TILING WITH FENCES

In order to generate a triangle we need to use exactly two types of tiles. To this end, we introduce a fence tile as a domino with the 'middle square' removed. In other words, one fence tile consists of two pieces, a left fence post and a right fence post. If we place a fence on cells i and $i + 1$, we cover the left-hand half of cell i and the right-hand half of cell $i + 1$ (see Figure 1(a)). Alternatively, a fence tile might straddle a cell as in Figure 1(b). Note that the total area covered by one fence equals that of one cell.

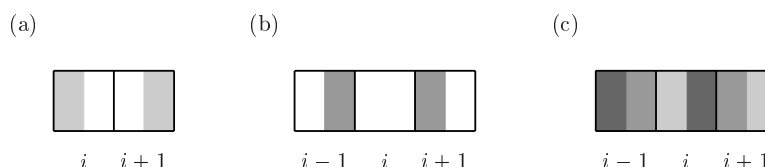


Figure 1.

Varieties of fence tilings: (a) non-straddling fence (b) straddling fence (c) trifence.

When tiling an n -board with squares and fences, what kind of configurations can arise? If we have exactly one non-straddling fence and no other fences, then we must tile the interior of the fence with a square. We refer to such a square as a captured square, and a fence along with its captured square as a filled fence. As with a domino, a filled fence can tile any two adjacent cells of an n -board, and is our first example of a metatile. A metatile is the smallest combination of squares and/or fences which completely tiles a minimum number of adjacent cells.

If a square tiles a cell, it is called a free square. A free square is a metatile, since it by itself completely covers one cell. A captured square is not a metatile, since it does not completely cover any single cell.

How many fences do we need to combine in order to make a metatile? One can easily verify that if we tile with interlocking straddling fences only, we will eventually be left with only half or one and a half squares to tile, which is impossible. Allowing the further use of squares leads to the same problem. Hence, if we have a straddling fence, we must also have a non-straddling fence. Having a straddling fence with an interlocking non-straddling fence, we are forced to add another non-straddling fence to produce what we will refer to as a trifence, pictured in Figure 1(c). In n -tilings, a trifence occupies the same configuration of cells as a triomino. Thus, the third and last metatile is the trifence.

The three metatiles introduced above will be used in counting arguments in the proofs of various identities. At this point we should emphasize that any tiling of an n -board using squares and fences can be created using squares, filled fences and trifences. In other words, there are only two ways in which fences appear in an n -tiling, namely, as part of a filled fence or as parts of a trifence. With this in mind, the proof of the following theorem is obvious.

Theorem 1. *Let s_n be the number of ways to tile an n -board with squares and fences. Then $s_n = t_n$.*

3. A PASCAL-LIKE TRIANGLE

Having only two tile types, squares and fences, we make the following definition. For $n \geq 1$, let the number $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ be the number of boards containing n tiles of which k are fences (and $n - k$ are squares). We remind the reader that replacing the word fence with domino in the above definition leads to a combinatorial interpretation of the binomial coefficient $\binom{n}{k}$. Note that since a fence covers the same area as a square, any tiling with k fences and $n - k$ squares tiles exactly n cells, that is, the length of the board equals the number of tiles used. Hence an alternative definition for $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ is the following. For $n \geq 1$, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ counts the number of n -tilings which contain exactly k fences.

We introduce a Pascal-like triangle as follows. Let the entry in the n th row and k th column of the triangle be $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$. For example, to find $\left[\begin{smallmatrix} 6 \\ 3 \end{smallmatrix} \right]$, consider a 6-board, to be tiled using exactly 3 fences. There is only one way to use 3 filled fences, whereas there are 4 ways to use a trifence, hence $\left[\begin{smallmatrix} 6 \\ 3 \end{smallmatrix} \right] = 5$. We define $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] = 1$ for convenience. Sometimes this is justified by saying that the empty board has one tiling, the empty tiling. The first ten rows of the resulting Pascal-like triangle are displayed in Figure 2. General strategies for calculating entries are given in the next section.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	0								
2	1	1	0							
3	1	2	0	1						
4	1	3	1	2	0					
5	1	4	3	3	2	0				
6	1	5	6	5	6	0	1			
7	1	6	10	9	12	3	3	0		
8	1	7	15	16	21	12	6	3	0	
9	1	8	21	27	35	30	14	12	0	1

Figure 2.
A Pascal-like triangle with entries $\begin{bmatrix} n \\ k \end{bmatrix}$.

4. PROOF OF IDENTITIES

Several identities will be proven some of which relate certain entries of the Pascal-like triangle to the triangular and tetrahedral numbers. In some proofs we will count fences, in others we will count squares. Bijections are also used. Finally, we will count both fences and squares simultaneously to prove an identity for a general entry in the triangle which reveals a connection with the binomial coefficients.

Identity 1. For $n \geq 0$,

$$T_{n+1} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}.$$

Proof. Note that the left-hand side equals t_n , as does the right-hand side by virtue of Theorem 1. □

Identity 2. For $n \geq 4$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-2 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-3 \\ k-3 \end{bmatrix}. \tag{1}$$

Proof. We count the number of n -tilings which contain exactly k fences given the tiling of the last cell of the n -board, cell n . If we tile the last cell with a square, there are $\begin{bmatrix} n-1 \\ k \end{bmatrix}$ ways to tile the first $(n-1)$ cells using k fences. If we tile the last cell with a fence, we have two possibilities. If it is a filled fence, we have tiled the last two cells, and there are $\begin{bmatrix} n-2 \\ k-1 \end{bmatrix}$ ways to tile the first $(n-2)$ cells using an additional $k-1$ fences. If it is a trifence, then we have tiled the last 3 cells, and there are $\begin{bmatrix} n-3 \\ k-3 \end{bmatrix}$ ways to tile the first $n-3$ cells using an additional $k-3$ fences. Adding these together gives the number of all n -tilings using exactly k fences. □

Note: One can easily see that if we define $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ whenever $n < 0$ or $k < 0$ or $k > n$ then the recurrence (1) continues to hold for all values of n and k except $n = k = 0$.

Identity 3. For $n \geq 1$,

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = n - 1.$$

Proof. Having only one fence, it must be part of a filled fence, which can start on any of the first $n - 1$ cells. \square

Most of the remaining identities involve triangular or tetrahedral numbers for which we introduce the following notation. For $n \geq 1$, we will denote the n th triangular number and the n th tetrahedral number by $G(n)$ and $H(n)$, respectively. We remind the reader that

$$G(n) = \frac{n(n+1)}{2} \quad \text{and} \quad H(n) = \frac{n(n+1)(n+2)}{6}.$$

Identity 4. For $n \geq 3$,

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = G(n - 3).$$

Proof. First note that two fences can occur only if they are both filled fences. Then, of the $n - 2$ squares in the tiling, two are captured and $n - 4$ are free, so this tiling has a total of $n - 2$ metatile positions. We must choose two of these positions to be filled fences. Hence there are

$$\binom{n-2}{2} = \frac{(n-2)(n-3)}{2} = G(n-3)$$

n -tilings which contain exactly two fences. \square

Identity 5. For $n \geq 5$,

$$\begin{bmatrix} n \\ 3 \end{bmatrix} = n - 2 + H(n - 5).$$

Proof. Note first that three fences can occur either as three filled fences or one trifence. Suppose the tiling contains three filled fences. Then of the $n - 3$ squares in the tiling, three are captured and $n - 6$ are free. Thus, this tiling has a total of $n - 3$ metatile positions. Choosing three of these positions to be filled fences, we have

$$\binom{n-3}{3} = \frac{(n-3)(n-4)(n-5)}{6} = H(n-5) \tag{2}$$

n -tilings which use three fences in the form of filled fences.

Now, suppose the tiling contains one trifence. Then the $n - 3$ squares in the tiling are all free, so this tiling has a total of $n - 2$ metatile positions, one of which we must choose to be a trifence. Hence, we have

$$\binom{n-2}{1} = n - 2 \tag{3}$$

n -tilings containing one trifence. Summing (2) and (3) proves the identity. \square

Identity 6. For $n \geq 1$,

$$\left[\begin{matrix} n \\ n-1 \end{matrix} \right] = \begin{cases} 0, & n \equiv 0 \pmod{3}, \\ \frac{1}{3}(n+2), & n \equiv 1 \pmod{3}, \\ \frac{1}{3}(n+1), & n \equiv 2 \pmod{3}. \end{cases}$$

Proof. Clearly this tiling contains exactly one square. That square can either be free or captured. If the square is free, then we are left with $n - 1$ cells to be tiled by say, $m - 1$ trifences. Thus $n - 1 = 3(m - 1)$, and hence $n = 3m - 2$ is the length of the board. Having one free square and $m - 1$ trifences, the total number of metatile positions is m . We need to choose one position for the free square. Hence the number of $(3m - 2)$ -tilings which contain exactly one free square is $\binom{m}{1} = m = \frac{1}{3}(n + 2)$.

If the square is captured, then we have a filled fence, and $n - 2$ cells to be tiled by $m - 1$ trifences. An analogous argument shows the number of tilings is $\frac{1}{3}(n + 1)$.

Finally, if the length of the board is $n = 3m$, then we have no tilings with exactly one square, since we have exhausted all possibilities in the above two cases. \square

Identity 7. For $n \geq 2$,

$$\left[\begin{matrix} n \\ n-2 \end{matrix} \right] = \begin{cases} 2G\left(\frac{n}{3}\right), & n \equiv 0 \pmod{3}, \\ G\left(\frac{n-1}{3}\right), & n \equiv 1 \pmod{3}, \\ G\left(\frac{n+1}{3}\right), & n \equiv 2 \pmod{3}. \end{cases}$$

Proof. This tiling contains exactly two squares. Suppose that both squares are free. We are left with $n - 2$ cells to be tiled with $m - 1$ trifences. Hence the board has length $n = 3m - 1$. We have $m + 1$ metatile positions, from which to choose two for the two free squares. Hence we have

$$\binom{m+1}{2} = \frac{m(m+1)}{2} = G(m) = G\left(\frac{n-1}{3}\right) \tag{4}$$

$(3m - 1)$ -tilings.

Suppose that both squares are captured. Then we are left with $n - 4$ cells to be tiled by $m - 1$ trifences. Hence the length of the board is $n = 3m + 1$, and there is a total of $m + 1$ metatile positions from which to choose two for the two filled fences. Again the number of $(3m + 1)$ -tilings which contain exactly two captured squares is $G(m) = G\left(\frac{n-1}{3}\right)$.

Lastly, suppose one square is free and the other is captured. Then we are left with $n - 3$ cells to be tiled by $m - 1$ trifences, so $n = 3m$. There are $m + 1$ metatile positions from which to choose two for the square and filled fence. Since either of these two positions can be the filled fence, the total number of ways is $2G(m) = 2G\left(\frac{n}{3}\right)$. \square

Lemma 1. For $m \geq 1$,

$$\left[\begin{matrix} 3m+1 \\ 3m-2 \end{matrix} \right] = \left[\begin{matrix} 3m+2 \\ 3m-1 \end{matrix} \right].$$

Proof. First note that any $(3m + 1)$ -tiling containing exactly three squares must contain two free squares and one filled fence. On the other hand, any $(3m + 2)$ -tiling containing exactly three squares must contain one free square and two filled fences. Consider the following bijection. Replace each of the two free squares in the $(3m + 1)$ -tiling by filled fences and replace the filled fence by a square. Leave all trifences the same. Note that we have gained one cell, that is, we now have a $(3m + 2)$ -tiling which contains exactly one free square and

two filled fences. Obviously, the procedure is reversible, we have a bijection, and hence the lemma holds. \square

Identity 8. For $n \geq 3$,

$$\begin{bmatrix} n \\ n-3 \end{bmatrix} = \begin{cases} H\left(\frac{n}{3}\right) + H\left(\frac{n-3}{3}\right), & n \equiv 0 \pmod{3}, \\ 3H\left(\frac{n-1}{3}\right), & n \equiv 1 \pmod{3}, \\ 3H\left(\frac{n-2}{3}\right), & n \equiv 2 \pmod{3}. \end{cases}$$

Proof. The tiling contains exactly three squares. A $3m$ -tiling can arise in two ways. Firstly, suppose all three squares are free. Then we are left with $n - 3$ cells to be tiled by $m - 1$ trifences and so the length of the board is $n = 3m$. Having three free squares and $m - 1$ trifences, we have $m + 2$ metatiles altogether. We choose three of the $m + 2$ positions to place the free squares. The number of ways to do this is

$$\binom{m+2}{3} = H(m) = H\left(\frac{n}{3}\right) \tag{5}$$

which counts the number of tilings with exactly three free squares.

Suppose all three squares are captured. Given a three free square $3(m - 1)$ -tiling, we can replace each free square by a captured square to get a three captured square $3m$ -tiling. This is clearly a bijection, hence, using (5), the number of three captured square $3m$ -tilings is $H(m - 1) = H\left(\frac{n-3}{3}\right)$.

Now, suppose we have two free squares and one captured square, so $n \equiv 1 \pmod{3}$, that is $n = 3m + 1$. Given a three free square tiling, we could replace any one of the free squares with a captured square to get a tiling with two free squares and one captured square. Thus there are three times as many $(3m + 1)$ -tilings which use two free squares and one captured square as there are $3m$ -tilings which use three free squares. Using (5) we have

$$\begin{bmatrix} 3m+1 \\ 3m-2 \end{bmatrix} = 3H(m) = 3H\left(\frac{n-1}{3}\right).$$

Similarly, if we have one free and two captured squares, $n \equiv 2 \pmod{3}$ and $n = 3m + 2$, and using Lemma 1, we obtain

$$\begin{bmatrix} 3m+2 \\ 3m-1 \end{bmatrix} = 3H(m) = 3H\left(\frac{n-2}{3}\right).$$

\square

Identity 9. For $0 \leq p \leq j \leq n$,

$$\begin{bmatrix} n \\ n-j \end{bmatrix} = \sum_{s=0}^N \binom{j}{p+3s} \binom{q+j-s}{j}$$

and if $0 \leq j < p \leq n$, then

$$\begin{bmatrix} n \\ n-j \end{bmatrix} = 0,$$

where $p = (n - j) \bmod 3$, $q = (n - j - p)/3$, and $N = \min(q, \lfloor (j - p)/3 \rfloor)$.

Proof. First observe that p is the minimum number of filled fences and q is the maximum number of trifences we can use in this n -tiling. In this minimum filled fence tiling, we have $j - p$ free squares and the number of metatile positions is $q + j$. To produce another tiling

with $n - j$ fences, we can convert one trifence into three filled fences after which the new tiling will contain $q - 1$ trifences, $p + 3$ filled fences, and $j - p - 3$ free squares. Hence, the number of metatile positions is now $q + j - 1$.

Similarly, after converting s trifences into $3s$ filled fences, the tiling will contain $q - s$ trifences, $p + 3s$ filled fences, and $j - p - 3s$ free squares since $3s$ free squares have been captured by the new filled fences. The number of metatile positions has been reduced to $q + j - s$. Let us count the number of tilings at this stage. We need to fill $q + j - s$ metatile positions. First, choose j of the $q + j - s$ metatile positions for placing squares. This can be done in $\binom{q+j-s}{j}$ ways. Now, in the j square positions choose $p + 3s$ places for the captured squares (filled fences). This can be done in $\binom{j}{p+3s}$ ways. Thus we have a total of $\binom{j}{p+3s} \binom{q+j-s}{j}$ ways to tile with $n - j$ fences, where $q - s$ are trifences.

Next, observe that we can continue to convert trifences into filled fences until we either run out of trifences or there are no free squares left to be captured by new filled fences. Hence we must have $s \leq q$ and $p + 3s \leq j$, that is, $s \leq q$ and $s \leq \lfloor (j - p)/3 \rfloor$.

To complete the proof we must check whether or not a minimum filled fence tiling always exists. Recall that we calculate $p = (n - j) \bmod 3$ to determine the minimum number of filled fences required. However, for each filled fence, a captured square is required. Thus we must have $j \geq p$. Otherwise, no tiling exists. \square

Note: This result combined with Identity 1 allows one to express the Tribonacci numbers in terms of sums of products of the binomial coefficients.

5. SUMS OF DIAGONAL ENTRIES

Consider the sequence defined by $U_0 = U_1 = U_2 = 1$, $U_3 = 2$, $U_4 = 3$, $U_5 = 4$, and for $n \geq 6$, $U_n = U_{n-1} + U_{n-3} + U_{n-6}$. From [2], U_n counts the number of ways to tile an n -board with squares, triominoes, and hexominoes (which we will call trihex n -tilings). Now, consider the Pascal-like triangle introduced in Section 3. The entries in the n th upward sloping diagonal are given by $\lfloor \binom{n-k}{k} \rfloor$ for $0 \leq k < \lfloor n/2 \rfloor$. Then the U_n 's can be expressed in terms of numbers of fence tilings as follows.

Theorem 2. For $n \geq 0$,

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \left\lfloor \binom{n-k}{k} \right\rfloor = U_n.$$

Proof. Consider the following bijection between trihex n -tilings and a certain set of fence tilings. Given a trihex n -tiling, produce a fence tiling by leaving square tiles the same, replacing all triominoes by filled fences, and replacing all hexominoes by trifences (see Figure 3). Observe that for each fence gained, one cell of the n -board is lost. Therefore, if k fences appear in the fence tiling, then the board will have length $n - k$. Finally note that $k \leq n - k$ if and only if $k \leq \lfloor n/2 \rfloor$. \square

Note: U_n is the sequence 1, 1, 2, 3, 4, 7, 11, 16, 25, 39, 59, 91, ... which is sequence A120415 in Sloane's Encyclopedia of Integer Sequences [7]. The above result appears to be a new interpretation of this sequence.

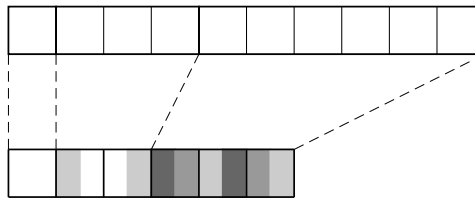


Figure 3.
Bijection for Theorem 2.

6. CONCLUSION

The results of this paper can be extended in various ways. The concept of fence tiling can reduce many third-order recurrences to enumerations of tilings using only two types of tiles, from which Pascal-like triangles can be constructed. For instance, we have already obtained similar results, to be presented elsewhere, for a triangle based on the generalized Tribonacci sequence using this method. In addition, different size fences can be constructed and fence-tiling arguments can be used to count the numbers of various types of tilings leading to new combinatorial identities. For example, take a larger fence tile consisting of the two square ends of a triomino. With this larger fence tile one can investigate properties of the sequence generated by the recurrence $a(n) = a(n-1) + a(n-3) + a(n-4)$ listed as sequence A006498 in Sloane's Encyclopedia of Integer Sequences [7]. This sequence has several connections with the Fibonacci numbers.

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